Semiparametric Conditional Quantile Models for Financial Returns and Realized Volatility

Jozef Barunik[⋆] and Filip Zikes[‡]

[⋆]Institute of Economic Studies, Faculty of Social Sciences, Charles University in Prague

[‡]The views expressed in this paper are those of the authors, and not necessarily those of the Bank of England.

DYME 2014 Wokshop
June 2, 2014

Prologue

- Financial econometrics literature focuses on measuring, modeling and forecasting of volatility
- Yet, a number of important decisions require estimation of entire distribution of future returns
- Desirability of conditional risk measurement (VaR):
  1. Portfolio-level (aggregated): Risk measurement
  2. Asset-level (disaggregated): Risk management

  $\Rightarrow$ Need for conditional density forecasts of returns
- Many situations with exposition to volatility risk
  $\Rightarrow$ Need for conditional density forecasts of volatility

Prologue

Prominent problems in finance calling for understanding entire distribution of returns

- Risk measurement and management (VaR)
- Portfolio allocation (when returns are non-Gaussian)
- Hedging and market timing strategies
- Derivative pricing

$\Rightarrow$ require a full characterisation of the multi-period forecasts of returns density, which will be conditional

Available literature

Only small part of literature pays attention to modeling conditional distribution. There are basically two strands:

- Andersen, Bollerslev, Diebold & Labys (2003), Giot & Laurent (2004) & Clements, Galvao & Kim (2008).
  $\Rightarrow$ Combine time series models for realized volatility with estimators of conditional distributions
- Brownlees & Gallo (2009), Shephard & Sheppard (2009) and Maheu & McCurdy (2010)
  $\Rightarrow$ Base predictive density on parametric return-based volatility models.
- Engle & Manganelli (2004) propose quantile regressions - CAViaR models

This paper

An important question

- Can we fully use high frequency data (and realized measures) in density forecasting?
  We provide answer!

- We take a different route
- We propose to couple the flexible quantile regression with realized measures
  $\Rightarrow$ Semiparametric framework with nonparametric measures
- ... and model conditional quantiles of daily returns and realized volatility
- To the best of our knowledge, quantile regressions has not been applied with realized volatilities

Coupling quantile regressions with realized volatility

In a simple way, we model and forecast the quantiles of distributions very successfully

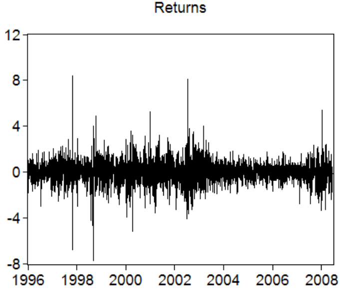
- avoiding restrictive assumptions on underlying conditional densities $\Leftarrow$ use of nonparametric measures
- understating information jumps carry about quantiles of future returns and volatility $\Leftarrow$ decomposition of ex-post variation to continuous and jump part
- understanding asymmetries $\Leftarrow$ use of realized semivariances
- capturing persistent dynamics through realized volatility

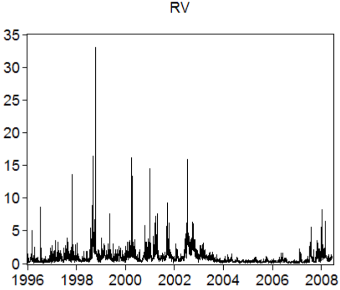
Main contributions of this paper

- We propose a simple model for conditional quantiles of returns.
- We propose heterogenous quantile autoregressions (HQAR) with jumps and implied volatility (extension of popular Corsi (2009))
- We introduce Realized CAViaR (enhance CAViaR model with realized measures)
- Models utilize realized measures decoupling jumps and continuous part of volatility
- Models utilize option implied volatility
- Models allow for asymmetries
- Our models forecast quantiles very precisely $h = 1, 5, 10$ steps ahead.

What's in the Data?

Returns and Realized Volatility of S&P 500





- Retruns: Approximately serially uncorrelated
- Not Gaussian (heavy-tailed)
- Heteroskedastic

Most important information: Volatility (translates to risk)

- Hence it is natural to think about quantiles as a function of volatility

Simple, but (can be) very missleading

Conditional dynamics is nowadays widely modeled by

- (G)ARCH models (and its many extensions)
- High Frequency data (non-parametric measure)
- Or recently, their combination (Realized GARCH of Hansen et al. (2011), HEAVY models by Shepard and Sheppard (2011))

These approaches are

- Fully parametric
- VaR is obtained simply by rescaling the distribution with volatility

Conditional quantiles modeling

As mentioned earlier,

- Very few comprehensive models for conditional distribution of returns
- No models for modeling distribution of volatility!

Our simple approach is to:

- Quantile regress returns on past realized volatility
- Qunatile regress realized volatility on past realized volatility

Quantile regression for returns

- A simple linear quantile regression can be used to estimate following model of conditional quantiles of returns:

$$
\begin{aligned}
q_\alpha(r_{t+1}|\Omega_t) &= \beta_0(\alpha) + \boldsymbol{\beta}_v(\alpha)'\boldsymbol{v}_t + \boldsymbol{\beta}_z(\alpha)'\boldsymbol{z}_t \\
r_{t+1} &= X_{t+1} - X_t \\
\boldsymbol{v}_t &= (QV_t^{1/2}, QV_{t-1}^{1/2}, \ldots, IV_t^{1/2}, IV_{t-1}^{1/2}, \ldots, \\
&\quad JV_t^{1/2}, JV_{t-1}^{1/2}, \ldots, VIX_t^{1/2}, VIX_{t-1}^{1/2})'
\end{aligned}
$$

- with $\boldsymbol{z}_t$ vector of weakly exogenous variables and $\boldsymbol{\beta}_j(\alpha)$ vectors of coefficients.

Quantile regression for returns cont.

- The parameters can be estimated by minimizing the following objective function

$$QR_{T,M}(\boldsymbol{\beta}(\alpha)) \equiv \frac{1}{T}\sum_{t=1}^{T}\rho_\alpha(r_{t+1}-\beta_0(\alpha)-\boldsymbol{\beta}_v(\alpha)'\boldsymbol{v}_{t,M}-\boldsymbol{\beta}_z(\alpha)'\boldsymbol{z}_t), \tag{1}$$

  where $\rho_\alpha(x) = (\alpha - \mathbf{1}\{x < 0\})x$, and $\boldsymbol{\beta}(\alpha) = (\beta_0(\alpha), \boldsymbol{\beta}_v(\alpha)', \boldsymbol{\beta}_z(\alpha)')'$.

Using different measures in $\boldsymbol{v}_t$, we can build several models

- decoupling IV and jumps
- realized semivariance
- Volatility implied by options (stock market's expectation)

Quantile regression for Volatility

- In addition, we propose a model for conditional quantiles for realized volatility.
- We call it heterogeneous quantile autoregressive model (HQAR) with jumps and implied volatility.
- As an extension of popular Corsi(2009).

Quantile regression for Volatility cont.

- As Realized volatility is observable measure of volatility, we can utilize this framework and forecast its distribution

$$
\begin{aligned}
q_\alpha(QV_{t+1}|\Omega_t) &= \beta_0(\alpha) + \boldsymbol{\beta}_{v1}(\alpha)'\boldsymbol{v}_t + \boldsymbol{\beta}_{v5}(\alpha)'\boldsymbol{v}_{t,t-5} + \\
&\quad + \boldsymbol{\beta}_{v22}(\alpha)'\boldsymbol{v}_{t,t-22} + \boldsymbol{\beta}_z(\alpha)'\boldsymbol{z}_t \\
\boldsymbol{v}_{t,t-k} &= 1/k \sum_{j=0}^{k-1} \boldsymbol{v}_{t-j}
\end{aligned}
$$

- with the same $\boldsymbol{v}_t$ (simplest is choice of QV_t)
- Note that this is (quantile) generalization of HAR
- Estimation reduces to Quantile Autoregression

Measurement error problem

- We need to make sure that impact of the measurement error vanishes so asymptotically, we obtain quantiles of true quadratic variation (and not realized variance measure)
- We provide sufficient conditions for convergence of infeasible objective function $\widehat{QR}_{T,M}$ to feasible QR_T
- Under the mild conditions A1-A3 (zero drift of the process and existing moments of the measurement error), we show that the number of intraday observations M has to grow faster than power of T, so the measurement error associated with realized measures and jump degenerate in limit.

Proposition 1

Under assumptions (A1) - (A3), if $T^{\frac{2}{2k-1}} M^{-1/2} \to 0$ as $T, M \to \infty$ and Θ is a compact parameter space, then $\sup_{\boldsymbol{\beta} \in \Theta} |QR_{T,M}(\boldsymbol{\beta}) - QR_T(\boldsymbol{\beta})| \xrightarrow{p} 0$.

Competing models: Realized CAViaR

To assess the relative performance of our approach, we consider:

- CAViaR model proposed by Engle and Manganelli (2004): Conditional Autoregressive Value at Risk by Regression Quantiles
- We augment it using various realized measures $\boldsymbol{x}_{t-1}$
- For daily asset return quantiles, $q_t(\boldsymbol{\theta})$, we consider following two specifications:
- Symmetric absolute value:

$$q_{t+1}(\boldsymbol{\theta}) = \beta_1 + \beta_2 q_t(\boldsymbol{\theta}) + \beta_3 |r_t| + \boldsymbol{\gamma}' \boldsymbol{x}_t, \tag{2}$$

- Asymmetric slope:

$$q_{t+1}(\boldsymbol{\theta}) = \beta_1 + \beta_2 q_t(\boldsymbol{\theta}) + \beta_3 (r_t)^+ + \beta_4 (r_t)^- + \boldsymbol{\gamma}' \boldsymbol{x}_{t-1}, \tag{3}$$

Competing models: Realized CAViaR

- Thus another novelty of our paper is to propose Realized CAViaR
- Utilize various realized measures again
- We are the first to test multi-day forecasts ($h = 1, 5, 10$) from CAViaR framework.
- We use direct forecasting.

Competing models: ARFIMA

- Second benchmark, we use long-memory lognormal-normal mixture proposed by proposed by Andersen et al. (2003)

$$
\begin{aligned}
r_t &= RV_{t,M}^{-1/2}\epsilon_t, && (4) \\
(1-\phi L)(1-L)^d \log RV_{t,M} &= (1-\psi L)u_t && (5)
\end{aligned}
$$

- $h = 1$ step ahead forecasts are available analytically
- We use simulations for multi-day forecasts ($h = 5, 10$)

Evaluation of quantile forecasts

- We evaluate *absolute* performance using DQ test of Engle & Manganelli (2004).
- Hit $Hit_{t+1} = \mathbf{1}\{r_{t+1} \leq q_\alpha(r_{t+1}|\Omega_t)\}$,
- Is regressed using logistic regression on its own past values.
- To assess *relative* performance, we use loss function of Giacomini & Komujer (2005):
- $L_\alpha(e_{t+1}) = (\alpha - 1\{e_{t+1} < 0\})e_{t+1}$,
- where $e_{t+1} = r_{t+1} - q_\alpha(r_{t+1}|\Omega_t)$.
- Equal predictive ability is tested with Diebold & Mariano (1995) test with the Newey-West correction for $h-$step ahead quantile forecasts.

Empirical Results

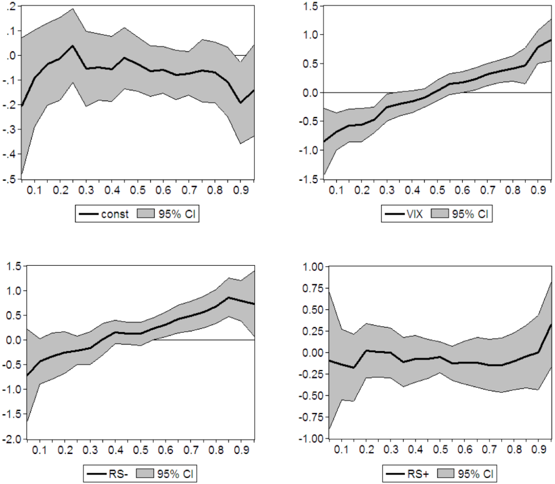
- S&P 500 and WTI Crude Oil 1996 - 2008
- We report only 5%, 10%, 50%, 90%, and 95%, but we can obtain any desired number of quantiles.
- $h = 1, 5, 10$ step ahead forecasts of all models.
- We model returns distributions and realized volatility distributions.
- Use realized volatility, MedRV, Jumps, RS^+, RS^- as regressors.
- We use VIX and construct MFIV for crude oil (OVX available only from 2007)

Results: Returns of S&P500

	Linear quantile regressions					CAViaR					Realized CAViaR				
α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
	LQR1					SAV					RSAV1				
const	−0.59 (−5.22)	−0.41 (−3.95)	−0.02 (−0.53)	0.11 (1.67)	0.24 (2.56)	−0.06 (−7.03)	−0.05 (−8.85)	0.00 (0.00)	0.02 (5.42)	0.04 (2.60)	0.00 (0.19)	−0.01 (−0.72)	−0.00 (−0.01)	−0.00 (−0.02)	0.02 (1.05)
q_t						0.92 (34.35)	0.91 (75.45)	0.30 (0.41)	0.97 (119.31)	0.96 (56.73)	0.80 (13.80)	0.83 (24.15)	0.29 (0.41)	0.77 (14.63)	0.81 (26.80)
$\|r_t\|$						−0.17 (−3.11)	−0.15 (−7.57)	−0.07 (−2.02)	0.05 (4.33)	0.07 (3.96)	0.02 (0.52)	−0.06 (−1.16)	−0.07 (−1.58)	−0.17 (−3.43)	−0.20 (−3.74)
$RV_t^{1/2}$	−1.12 (−9.52)	−0.88 (−7.24)	0.05 (1.18)	1.15 (11.9)	1.43 (12.0)						−0.34 (−2.52)	−0.19 (−2.14)	0.00 (0.01)	0.37 (5.76)	0.38 (6.68)
	LQR2										RSAV2				
const	−0.16 (−1.18)	−0.11 (−1.17)	−0.05 (−0.93)	−0.19 (−2.23)	−0.17 (−2.26)						0.18 (2.74)	0.14 (2.21)	0.05 (0.49)	−0.00 (−0.13)	0.14 (0.90)
q_t											0.07 (0.43)	0.24 (0.97)	−0.75 (−2.22)	0.76 (9.73)	0.18 (0.71)
$\|r_t\|$											0.04 (0.93)	0.02 (0.28)	−0.01 (−0.96)	−0.16 (−3.01)	−0.23 (−2.69)
$IV_t^{1/2}$	−0.48 (−2.09)	−0.40 (−2.60)	0.03 (0.54)	0.53 (3.55)	0.67 (3.96)						−0.68 (−5.13)	−0.51 (−2.86)	0.06 (1.10)	0.36 (3.82)	0.67 (1.98)
$JV_t^{1/2}$	−0.80 (−1.24)	0.10 (0.11)	0.11 (0.36)	0.00 (0.01)	0.04 (0.18)						−0.03 (−0.07)	0.02 (0.38)	−0.23 (−2.01)	0.05 (0.10)	0.80 (1.22)
VIX_t	−0.92 (−3.48)	−0.64 (−4.02)	0.05 (0.60)	0.79 (6.07)	0.99 (6.00)						−0.90 (−4.40)	−0.53 (−2.11)	−0.13 (−0.99)	0.02 (0.39)	0.58 (1.09)
	LQR3					AS					RAS				
const	−0.20 (−1.44)	−0.09 (−0.96)	−0.04 (−0.67)	−0.19 (−2.42)	−0.14 (−1.62)	−0.01 (−1.66)	−0.01 (−1.18)	0.01 (0.75)	0.01 (2.81)	0.02 (1.72)	0.07 (1.52)	0.01 (0.29)	−0.00 (−0.03)	0.04 (1.84)	0.04 (1.89)
q_t						0.94 (65.44)	0.93 (94.83)	0.74 (6.22)	0.97 (112.36)	0.97 (69.00)	0.35 (1.42)	0.85 (16.72)	0.70 (3.25)	0.80 (9.86)	0.85 (17.61)
$(r_t)^+$						−0.00 (−0.12)	−0.01 (−0.60)	0.04 (2.41)	−0.02 (−1.09)	−0.02 (−1.13)	0.20 (1.51)	0.07 (1.40)	0.10 (1.14)	−0.23 (−1.70)	−0.28 (−3.01)
$(r_t)^-$						−0.20 (−7.73)	−0.19 (−10.51)	−0.09 (−4.12)	0.08 (7.37)	0.11 (4.17)	0.05 (0.47)	−0.12 (−2.22)	−0.15 (−2.03)	−0.07 (−1.07)	−0.04 (−0.40)
${RS_t^+}^{1/2}$	−0.09 (−0.23)	−0.14 (−0.68)	−0.06 (−0.64)	−0.00 (−0.02)	0.31 (1.30)						−0.22 (−0.43)	−0.07 (−0.72)	−0.33 (−0.67)	0.36 (0.73)	0.29 (0.66)
${RS_t^-}^{1/2}$	−0.71 (−1.51)	−0.43 (−1.91)	0.12 (1.02)	0.78 (3.72)	0.73 (2.08)						−0.69 (−1.13)	−0.13 (−0.85)	0.30 (0.79)	0.09 (0.23)	0.13 (0.27)
VIX_t	−0.85 (−2.99)	−0.67 (−4.39)	0.03 (0.33)	0.78 (5.27)	0.90 (4.76)						−0.51 (−1.71)	−0.03 (−0.96)	0.02 (0.31)	0.02 (0.27)	0.04 (0.67)

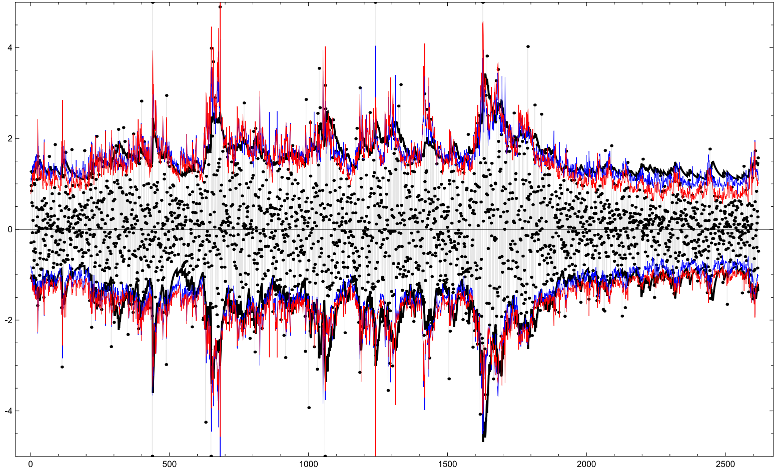
Results: Returns of S&P500

Graphical representation of estimated quantile regression process (LQR3 model)



Results: 1% and 99% quantiles S&P500

CAViaR in black, Realized CAViaR in blue, LQR in red



Results: Returns of WTI Crude Oil

	Linear quantile regressions					CAViaR					Realized CAViaR				
α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
	LQR1					SAV					RSAV1				
const	−1.59 (−3.55)	−1.37 (−4.74)	0.19 (0.86)	1.52 (6.24)	1.87 (4.23)	−0.04 (−2.18)	−0.02 (−1.50)	−0.29 (−3.03)	0.07 (1.52)	0.06 (1.21)	−0.03 (−0.43)	−0.00 (−0.07)	0.01 (0.17)	0.21 (1.28)	0.15 (1.00)
q_t						0.97 (108.83)	0.98 (76.79)	−0.95 (−13.87)	0.95 (26.10)	0.96 (33.79)	0.90 (14.85)	0.90 (13.92)	0.66 (4.53)	0.77 (5.06)	0.85 (7.49)
$\|r_t\|$						−0.06 (−2.90)	−0.03 (−2.66)	0.02 (0.99)	0.04 (1.49)	0.05 (1.70)	0.01 (0.11)	0.05 (1.11)	0.10 (3.04)	−0.02 (−0.41)	0.02 (0.26)
$RV_t^{1/2}$	−0.68 (−2.66)	−0.41 (−2.23)	−0.04 (−0.35)	0.40 (2.65)	0.59 (2.31)						−0.15 (−1.15)	−0.15 (−1.55)	−0.09 (−2.08)	0.16 (1.59)	0.15 (1.08)
	LQR2										RSAV2				
const	−0.68 (−1.47)	−0.40 (−0.96)	0.61 (1.85)	0.75 (3.02)	0.68 (1.14)						0.01 (0.16)	0.01 (0.26)	−0.93 (−3.36)	0.42 (1.40)	0.98 (1.43)
q_t											0.95 (36.30)	0.94 (26.19)	−0.63 (−4.54)	−0.03 (−0.06)	−0.73 (−2.92)
$\|r_t\|$											−0.01 (−0.14)	0.05 (1.66)	0.04 (1.57)	0.00 (0.05)	−0.06 (−0.89)
$IV_t^{1/2}$	0.08 (0.27)	0.14 (0.69)	0.17 (0.95)	−0.08 (−0.60)	0.30 (1.06)						−0.04 (−0.72)	−0.07 (−1.75)	−0.10 (−0.75)	−0.15 (−0.80)	0.33 (1.76)
$JV_t^{1/2}$	−0.48 (−1.46)	−0.63 (−1.27)	0.09 (0.25)	0.23 (0.74)	0.29 (0.49)						−0.42 (−2.66)	−0.34 (−3.06)	−0.66 (−10.35)	0.36 (1.28)	0.34 (2.32)
ImV_t	−1.06 (−3.60)	−0.95 (−3.82)	−0.39 (−1.94)	0.82 (4.94)	0.82 (2.22)						−0.03 (−1.16)	−0.02 (−0.90)	0.42 (2.88)	1.00 (1.78)	1.68 (3.34)
	LQR3					AS					RAS				
const	−0.54 (−1.06)	−0.45 (−1.28)	0.61 (2.04)	0.86 (2.92)	0.62 (1.01)	−0.01 (−0.60)	−0.01 (−0.49)	−0.29 (−2.78)	0.08 (1.12)	0.15 (1.54)	−0.37 (−1.15)	−0.73 (−2.59)	−0.62 (−2.43)	0.57 (1.08)	1.10 (1.04)
q_t						0.96 (78.54)	0.97 (41.91)	−0.19 (−0.86)	0.92 (16.46)	0.90 (16.43)	0.56 (2.98)	0.17 (1.00)	−0.20 (−0.80)	−0.10 (−0.15)	−0.94 (−27.22)
$(r_t)^+$						−0.05 (−1.55)	−0.03 (−0.96)	0.17 (2.01)	0.04 (0.93)	0.05 (0.71)	0.18 (1.13)	0.30 (4.45)	0.16 (2.10)	0.00 (0.00)	−0.04 (−0.66)
$(r_t)^-$						−0.09 (−3.02)	−0.06 (−1.84)	−0.07 (−1.28)	0.07 (1.54)	0.15 (1.88)	−0.10 (−0.77)	−0.27 (−2.18)	−0.03 (−0.36)	0.06 (0.36)	0.04 (0.68)
$RS_t^{+1/2}$	−0.12 (−0.31)	0.03 (0.13)	0.45 (2.79)	−0.35 (−1.29)	−0.23 (−0.83)						−0.06 (−0.13)	−0.57 (−3.81)	−0.03 (−0.12)	0.06 (0.12)	0.20 (0.99)
$RS_t^{-1/2}$	−0.01 (−0.02)	0.17 (0.72)	−0.39 (−2.07)	0.51 (1.80)	0.58 (2.12)						−0.12 (−0.27)	0.72 (3.36)	−0.15 (−0.69)	−0.28 (−0.54)	0.00 (0.00)
ImV_t	−0.99 (−3.57)	−0.95 (−3.82)	−0.28 (−1.38)	0.59 (2.65)	0.89 (2.86)						−0.36 (−1.42)	−0.69 (−3.90)	0.28 (1.51)	0.99 (1.42)	2.07 (4.09)

Results: Returns of S&P500

LQR models has exact conditional coverage and are well specified ($p<0.05$, model is misspecified)

		in-sample					out-of-sample				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.050	0.102	0.482	0.880	0.935	0.066	0.116	0.478	0.904	0.950
	DQ	1.831	2.177	16.92	27.26	16.53	7.829	7.745	4.274	12.89	8.488
	p-val	0.939	0.904	0.013	0.000	0.023	0.336	0.287	0.675	0.054	0.257
SAV	$\hat{\alpha}$	0.050	0.100	0.501	0.900	0.950	0.050	0.092	0.506	0.926	0.966
	DQ	5.217	8.195	12.14	20.39	16.16	5.789	7.104	4.169	22.94	7.006
	p-val	0.536	0.237	0.063	0.003	0.025	0.579	0.309	0.676	0.001	0.433
RSAV1	$\hat{\alpha}$	0.050	0.100	0.501	0.901	0.951	0.074	0.118	0.508	0.896	0.948
	DQ	2.048	3.748	10.31	18.61	3.658	6.113	2.581	5.014	15.59	10.25
	p-val	0.901	0.717	0.115	0.007	0.725	0.540	0.864	0.568	0.027	0.122
RSAV2	$\hat{\alpha}$	0.050	0.100	0.501	0.901	0.952	0.070	0.116	0.510	0.896	0.948
	DQ	6.033	3.762	11.03	18.43	3.225	5.805	2.711	3.181	6.246	9.235
	p-val	0.436	0.699	0.096	0.007	0.773	0.581	0.868	0.800	0.429	0.182
AS	$\hat{\alpha}$	0.050	0.100	0.502	0.900	0.949	0.054	0.094	0.488	0.910	0.952
	DQ	6.493	2.011	1.266	10.81	8.185	6.409	4.768	1.538	5.365	7.915
	p-val	0.402	0.911	0.970	0.103	0.239	0.512	0.601	0.951	0.517	0.323
RAS	$\hat{\alpha}$	0.050	0.101	0.501	0.900	0.951	0.054	0.102	0.496	0.908	0.958
	DQ	5.658	0.948	0.313	8.482	0.903	1.192	2.202	1.326	12.49	6.290
	p-val	0.463	0.990	1.000	0.225	0.989	0.987	0.902	0.975	0.058	0.545
LQR1	$\hat{\alpha}$	0.050	0.100	0.500	0.900	0.950	0.056	0.096	0.496	0.918	0.964
	DQ	6.183	3.063	14.43	15.14	8.434	4.515	6.179	3.225	9.359	7.750
	p-val	0.398	0.802	0.025	0.023	0.226	0.760	0.399	0.780	0.191	0.331
LQR2	$\hat{\alpha}$	0.050	0.100	0.500	0.900	0.950	0.056	0.104	0.490	0.902	0.954
	DQ	2.067	1.092	13.09	8.941	3.918	4.581	4.357	3.611	10.67	7.513
	p-val	0.929	0.983	0.037	0.176	0.681	0.744	0.616	0.721	0.126	0.393
LQR3	$\hat{\alpha}$	0.050	0.100	0.500	0.900	0.950	0.052	0.104	0.494	0.898	0.956
	DQ	4.688	1.255	12.13	10.31	3.497	3.578	7.499	3.319	6.723	6.576
	p-val	0.591	0.975	0.050	0.109	0.771	0.862	0.313	0.761	0.387	0.503

Results: Returns of S&P500 forecasts

		h=1					h=5					h=10				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.066	0.116	0.478	0.904	0.950	0.062	0.108	0.476	0.916	0.970	0.064	0.092	0.452	0.962	0.984
	$\hat{L}$	0.088	0.149	0.318	0.138	0.082	0.200	0.319	0.624	0.264	0.154	0.264	0.426	0.817	0.362	0.225
	DM	-4.608†	-4.819†	-0.769	0.593	-0.089	0.297	-0.148	-1.344	-0.098	1.267	0.976	0.324	-1.686†	0.891	0.787
SAV	$\hat{\alpha}$	0.050	0.092	0.506	0.926	0.966	0.046	0.094	0.496	0.924	0.966	0.034	0.078	0.448	0.950	0.972
	$\hat{L}$	0.103	0.164	0.318	0.144	0.091	0.200	0.330	0.627	0.272	0.162	0.254	0.428	0.825	0.353	0.222
	DM	2.514*	1.990*	-0.626	2.176*	2.733*	0.373	1.290	-0.869	0.937	2.793*	0.526	0.769	-1.495	0.306	0.738
RSAV1	$\hat{\alpha}$	0.074	0.118	0.508	0.896	0.948	0.052	0.084	0.490	0.888	0.958	0.056	0.092	0.434	0.910	0.962
	$\hat{L}$	0.098	0.159	0.318	0.141	0.084	0.203	0.324	0.632	0.266	0.153	0.280	0.435	0.837	0.361	0.222
	DM	0.796	-0.293	-0.304	2.047*	1.184	0.922	0.471	-0.018	0.351	1.510	1.837*	1.005	-0.789	1.047	1.262
RSAV2	$\hat{\alpha}$	0.070	0.116	0.510	0.896	0.948	0.052	0.090	0.490	0.882	0.950	0.054	0.082	0.454	0.916	0.960
	$\hat{L}$	0.097	0.159	0.318	0.139	0.082	0.201	0.321	0.633	0.268	0.152	0.267	0.410	0.827	0.359	0.223
	DM	0.590	-0.151	-0.279	3.224*	0.792	0.525	0.054	0.224	0.855	1.431	1.436	-1.383	-1.434	1.047	1.579
AS	$\hat{\alpha}$	0.054	0.094	0.488	0.910	0.952	0.048	0.088	0.492	0.928	0.960	0.036	0.074	0.456	0.932	0.976
	$\hat{L}$	0.101	0.161	0.316	0.139	0.084	0.195	0.326	0.622	0.260	0.152	0.259	0.428	0.843	0.347	0.207
	DM	1.965*	0.600	-1.589	0.902	1.163	-0.310	0.687	-1.595	-0.509	0.731	1.239	0.666	0.167	0.016	-0.280
RAS	$\hat{\alpha}$	0.054	0.102	0.496	0.908	0.958	0.050	0.082	0.494	0.884	0.946	0.048	0.086	0.456	0.908	0.956
	$\hat{L}$	0.098	0.157	0.316	0.135	0.083	0.200	0.324	0.629	0.267	0.151	0.265	0.415	0.851	0.363	0.218
	DM	0.814	-1.028	-1.659†	-0.877	0.425	0.516	0.536	-0.581	0.757	1.112	1.730*	-0.644	0.748	1.037	0.386
LQR1	$\hat{\alpha}$	0.056	0.096	0.496	0.918	0.964	0.050	0.078	0.484	0.900	0.958	0.046	0.072	0.436	0.928	0.970
	$\hat{L}$	0.098	0.161	0.317	0.137	0.083	0.203	0.329	0.631	0.262	0.152	0.266	0.434	0.838	0.347	0.210
	DM	1.472	1.111	-1.506	0.299	0.534	1.029	1.457	-0.897	-0.357	1.022	2.363*	2.116*	-0.886	-0.094	-0.281
LQR3	$\hat{\alpha}$	0.052	0.104	0.494	0.898	0.956	0.048	0.082	0.480	0.880	0.948	0.044	0.084	0.430	0.910	0.960
	$\hat{L}$	0.099	0.159	0.318	0.136	0.081	0.197	0.320	0.631	0.260	0.146	0.249	0.420	0.841	0.349	0.212
	DM	1.371	0.224	-0.747	-1.293	-0.769	0.244	-0.161	-0.915	-2.093†	-0.616	0.223	-0.487	-0.651	0.687	-0.722
LQR2	$\hat{\alpha}$	0.056	0.104	0.490	0.902	0.954	0.046	0.080	0.478	0.878	0.950	0.042	0.084	0.432	0.908	0.962
	$\hat{L}$	0.096	0.159	0.318	0.137	0.082	0.197	0.320	0.632	0.264	0.147	0.249	0.421	0.842	0.348	0.213

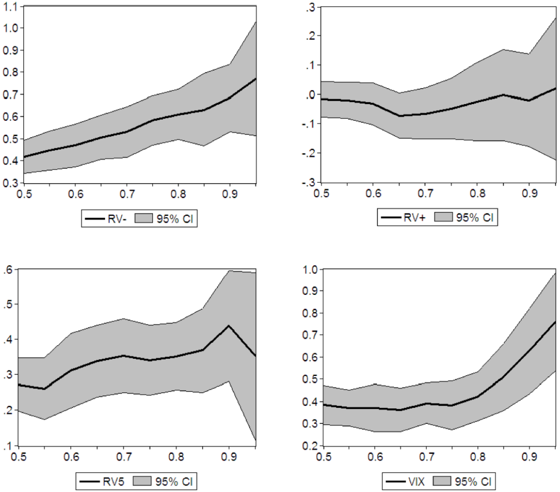
Results: Realized Volatility of S&P500

Note stable increase in $RV_t^{1/2}$ $\Rightarrow$ volatility-of-volatility effect observed by literature

	HARQ1				HARQ2				HARQ3			
α	0.50	0.75	0.90	0.95	0.50	0.75	0.90	0.95	0.50	0.75	0.90	0.95
A. Parameter estimates												
const	0.08 (5.55)	0.07 (4.23)	0.06 (2.77)	0.05 (1.31)	−0.01 (−0.70)	−0.01 (−0.46)	−0.07 (−2.84)	−0.11 (−2.68)	−0.02 (−1.32)	−0.04 (−1.91)	−0.10 (−3.36)	−0.15 (−3.71)
$RV_t^{1/2}$	0.37 (9.33)	0.50 (9.31)	0.70 (11.1)	0.78 (7.69)								
$RS_t^{+1/2}$					−0.01 (−0.59)	−0.05 (−0.93)	−0.02 (−0.26)	0.01 (0.15)				
$RS_t^{-1/2}$					0.41 (11.3)	0.58 (10.1)	0.68 (8.70)	0.76 (6.21)				
$RV_{t,t-5}^{1/2}$	0.32 (6.01)	0.40 (6.67)	0.43 (4.23)	0.49 (3.04)	0.27 (7.13)	0.34 (6.99)	0.43 (5.75)	0.35 (2.90)				
$RV_{t,t-22}^{1/2}$	0.16 (4.37)	0.12 (2.85)	0.11 (1.41)	0.12 (0.84)	−0.04 (−1.18)	−0.04 (−1.16)	−0.24 (−2.85)	−0.23 (−2.33)				
$IV_t^{1/2}$									0.30 (10.8)	0.43 (7.23)	0.53 (7.04)	0.63 (6.17)
$IV_{t,t-5}^{1/2}$									0.22 (5.44)	0.27 (4.76)	0.31 (4.29)	0.26 (2.16)
$IV_{t,t-22}^{1/2}$									−0.04 (−1.22)	−0.13 (−2.74)	−0.24 (−2.61)	−0.27 (−2.45)
$JV_t^{1/2}$									0.03 (0.50)	0.14 (1.40)	0.12 (0.60)	0.46 (1.82)
VIX_t					0.38 (9.03)	0.38 (6.94)	0.62 (6.49)	0.75 (6.93)	0.41 (10.6)	0.50 (9.01)	0.71 (6.89)	0.84 (7.39)

Results: Realized Volatility of S&P500

Note stable increase in $VIX \Rightarrow$ volatility-of-volatility dependent also on expectations



Results: Realized Volatility of S&P500

		in-sample				out-of-sample			
	α	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95
A. S&P 500									
ARFIMA	$\hat{\alpha}$	0.534	0.780	0.907	0.948	0.522	0.837	0.954	0.978
	DQ	25.08	29.13	24.53	33.08	46.73	51.52	31.01	15.67
	p-val	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.007
LQR1	$\hat{\alpha}$	0.500	0.750	0.900	0.949	0.554	0.770	0.894	0.944
	DQ	20.31	12.65	2.764	6.430	19.50	13.98	4.617	3.249
	p-val	0.002	0.045	0.820	0.405	0.002	0.028	0.613	0.859
LQR2	$\hat{\alpha}$	0.500	0.750	0.900	0.950	0.540	0.750	0.864	0.928
	DQ	18.93	25.25	5.369	4.877	12.99	6.462	10.09	6.633
	p-val	0.005	0.000	0.486	0.577	0.034	0.369	0.133	0.482
LQR3	$\hat{\alpha}$	0.500	0.750	0.900	0.950	0.538	0.750	0.862	0.928
	DQ	36.50	29.10	5.574	6.529	9.771	11.69	9.858	7.796
	p-val	0.000	0.000	0.475	0.368	0.134	0.068	0.178	0.334
B. WTI Crude Oil									
ARFIMA	$\hat{\alpha}$	0.553	0.781	0.892	0.937	0.510	0.825	0.948	0.974
	DQ	47.61	26.63	22.36	13.97	13.33	21.03	19.52	13.71
	p-val	0.000	0.000	0.003	0.040	0.039	0.002	0.003	0.028
LQR1	$\hat{\alpha}$	0.501	0.750	0.901	0.949	0.524	0.734	0.884	0.948
	DQ	8.308	8.639	0.935	4.332	7.497	5.594	1.929	8.149
	p-val	0.207	0.189	0.989	0.649	0.274	0.486	0.923	0.278
LQR2	$\hat{\alpha}$	0.501	0.750	0.900	0.950	0.520	0.736	0.890	0.946
	DQ	11.79	9.121	1.729	1.954	7.263	7.053	3.484	3.976
	p-val	0.070	0.165	0.952	0.941	0.299	0.325	0.733	0.790
LQR3	$\hat{\alpha}$	0.501	0.750	0.900	0.949	0.524	0.730	0.900	0.950
	DQ	11.68	9.313	1.593	2.162	9.959	5.950	5.085	12.80
	p-val	0.071	0.175	0.950	0.916	0.148	0.435	0.559	0.033

Results: Realized Volatility of S&P500

		h=1				h=5				h=10			
	α	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95
A. S&P 500													
ARFIMA	$\hat{\alpha}$	0.522	0.837	0.954	0.978	0.550	0.745	0.849	0.902	0.556	0.723	0.823	0.865
	$\hat{L}$	0.051	0.047	0.029	0.018	0.066	0.063	0.043	0.029	0.073	0.073	0.053	0.038
	DM	-15.66†	-10.84†	-6.093†	-4.286†	-1.852†	-0.391	0.003	-0.307	-0.257	0.648	0.911	1.025
LQR1	$\hat{\alpha}$	0.554	0.770	0.894	0.944	0.582	0.734	0.882	0.926	0.584	0.752	0.858	0.908
	$\hat{L}$	0.079	0.072	0.044	0.029	0.076	0.068	0.045	0.030	0.080	0.075	0.049	0.032
	DM	4.307*	2.437*	1.559	2.224*	4.606*	2.812*	1.580	1.227	2.645*	2.138*	1.079	0.999
LQR2	$\hat{\alpha}$	0.540	0.750	0.864	0.928	0.536	0.732	0.870	0.910	0.558	0.740	0.848	0.898
	$\hat{L}$	0.074	0.068	0.042	0.027	0.069	0.064	0.042	0.029	0.074	0.070	0.047	0.031
	DM	-1.707	-1.228	-0.415	0.477	-1.796	-0.954	-0.578	-1.189	0.684	-0.651	-0.649	-0.583
LQr3	$\hat{\alpha}$	0.538	0.750	0.862	0.928	0.536	0.734	0.870	0.920	0.568	0.730	0.842	0.900
	$\hat{L}$	0.075	0.069	0.042	0.026	0.070	0.064	0.043	0.029	0.074	0.070	0.047	0.031
B: WTI Crude Oil													
ARFIMA	$\hat{\alpha}$	0.510	0.825	0.948	0.974	0.488	0.783	0.904	0.944	0.528	0.747	0.873	0.902
	$\hat{L}$	0.096	0.085	0.053	0.033	0.078	0.069	0.042	0.026	0.081	0.074	0.045	0.027
	DM	-9.949†	-7.823†	-5.159†	-3.908†	-1.518†	-0.148	0.719	0.486	0.349	1.089	1.036	0.649
LQR1	$\hat{\alpha}$	0.524	0.734	0.884	0.948	0.522	0.754	0.886	0.926	0.554	0.752	0.876	0.924
	$\hat{L}$	0.127	0.116	0.070	0.043	0.088	0.079	0.047	0.028	0.087	0.077	0.047	0.028
	DM	1.070	2.605*	2.426*	1.730*	1.119	2.285*	2.094*	1.473	2.118*	1.907*	1.622	1.057
LQR2	$\hat{\alpha}$	0.520	0.736	0.890	0.946	0.520	0.720	0.902	0.952	0.552	0.742	0.902	0.940
	$\hat{L}$	0.126	0.113	0.066	0.041	0.086	0.070	0.040	0.024	0.080	0.068	0.039	0.026
	DM	0.933	2.295	0.525	0.478	1.552	0.528	0.513	0.142	0.727	-0.386	-1.274	1.793
LQR3	$\hat{\alpha}$	0.524	0.730	0.900	0.950	0.516	0.728	0.898	0.950	0.548	0.742	0.894	0.944
	$\hat{L}$	0.126	0.111	0.066	0.040	0.085	0.070	0.040	0.024	0.079	0.068	0.039	0.025

Summary of empirical results

- Realized volatility and implied volatility carry significant information about future quantiles of returns as well as volatility.
  $\Rightarrow$ persistent volatility drives persisten quantiles
- Negative semivariance drives both left and right tail (positive does not have significant impact)
- Jumps play (although small) role in forecasts
- HARQ captures conditional quantiles of volatility well
- In a multi-day ahead forecasts, LQR and HARQ performs best, especially in a right tail of distribution.

Epilogue

- Yes, we can easily forecast full distribution of returns using high-frequency data
- Yes, we can easily forecast full distribution of volatility
- Improvement is greater in multi-day ahead forecasts
- We allow for reliable conditional risk measurement
- Future work: many extensions, one of them: use Quntile VAR.

I have provided preview of results: Full paper online

In case you are interested, paper is available through SSRN, or RePEc
(version after revision in JoFE)

Thank you very much for your attention!